

Parameter uncertainty and reserve risk under Solvency II

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Abstract

In this article we consider the parameter risk in the context of internal modelling of the reserve risk under Solvency II.

We discuss two opposed perspectives on parameter uncertainty and point out that standard methods of classical reserving focusing on the estimation error of claims reserves are in general not appropriate to model the impact of parameter uncertainty upon the actual risk of economic losses from the undertakings's perspective.

Referring to the requirements of Solvency II we assess methods to model parameter uncertainty for the reserve risk by comparing the probability of solvency actually attained when modelling the solvency risk capital requirement based on the respective method to the required confidence level. Using the simple example of a normal model we show that the bootstrapping approach is not appropriate to model parameter uncertainty according to this criterion. We then present an adaptation of the approach proposed in Fröhlich and Weng (2015). Experimental results demonstrate that this new method yields a risk capital model for the reserve risk achieving the required confidence level in good approximation.

Keywords: Solvency II, parameter uncertainty, reserving risk, Solvency capital, internal model

1 Introduction

The Solvency II directive (Solvency directive 2009/138/EC (2009)) defines the capital requirement of an insurance undertaking as the value-at-risk of

the loss of basic own funds for the confidence level $\alpha = 99.5\%$ over a one-year time horizon (cf. Solvency directive 2009/138/EC (2009) Article 101). We interpret the change in basic own funds as a random variable.

An effective risk management requires not only the consideration of the overall risk of an insurance undertaking, but also an assessment of the material subrisks. If we interpret the loss in basic own funds over a one-year horizon due to a particular subrisk as a random variable \mathbf{X} , it is best practice to define the standalone risk capital requirement for this subrisk analogously to Article 101 in Solvency directive 2009/138/EC (2009) as the 99.5% value-at-risk of \mathbf{X} .

However, there is not only uncertainty about the future outcomes of \mathbf{X} caused by random fluctuation, but also about the true distribution of \mathbf{X} . Therefore, the true 99.5% value-at-risk of \mathbf{X} is unknown and the insurance undertaking can only estimate its solvency capital requirement.

In this article, we assume that the undertaking uses an internal model and estimates the parameters specifying \mathbf{X} from historical data. The possible deviation of the parameter estimates from the true parameters causes parameter uncertainty. In the sequel we ignore the basic model uncertainty and concentrate on the parameter uncertainty.

In this situation we have two sources of uncertainty:

1. the random variable \mathbf{X} ,
2. the uncertainty with respect to the modelled solvency capital requirement **SCR** resulting from the randomness of the historical data used to estimate the parameters.

Note that due to 2. the modelled solvency capital requirement **SCR** itself is a random variable.

From Solvency directive 2009/138/EC (2009), Article 101 we derive the following question:

Question 1.1. How can we model the solvency capital requirement **SCR** (for a subrisk) such that it will not be exceeded by the possible loss \mathbf{X} of basic own funds (due to this subrisk) over a one-year horizon with a probability of 99.5% - taking into account the randomness of both, \mathbf{X} and the **SCR**?

This question corresponds to a central idea of predictive inference, see e.g. Barndorff-Nielsen and Cox (1996); Young and Smith (2005); Severini et al. (2002), and has been investigated in the context of Solvency II for several distributional assumptions for the random variable \mathbf{X} in various articles (see e.g. Gerrard and Tsanakas (2011); Fröhlich and Weng (2015); Bignozzi and Tsanakas (2016, 2015)).

In the sequel we restrict to the modelling of a standalone solvency capital requirement for the reserve risk in the sense of Question 1.1. In the context of the reserve risk, \mathbf{X} is the loss according to the one-year development result of incurred claims and the solvency capital requirement **SCR** is the standalone solvency capital requirement for the one-year reserve risk. We assume that we can write $\mathbf{X} = \mathbf{X}(\theta)$ for some fixed, but unknown parameter vector θ . The undertaking does not know θ but can only estimate $\hat{\theta}$ based on the observed claims development triangle D . These notions will be made precise in Section 2. For the sake of simplicity we ignore the impact of interest rates and (the development result of) the risk margin.

There is an extensive literature on the reserve risk dealing with the prediction error of claims reserves. However, the majority of these contributions does not consider the value-at-risk of the one-year claims development result, but investigates the ultimate mean squared prediction error, see e.g. Mack (1993, 1999); Wüthrich and Merz (2008). Merz and Wüthrich (2008) investigate the one-year development result, but they also use the mean squared estimation error as the risk measure.

We point out that the literature mentioned above discusses the mean squared estimation error with respect to the expected value of future claims (best estimate reserve) but provides no results concerning the value-at-risk (solvency capital requirement) of a predictive distribution taking the uncertainty about all parameters of the process distribution into account. In particular, for Mack's chain ladder model the uncertainty about the volatility parameters σ_k^2 is often ignored by just replacing σ_k^2 by its estimate $\hat{\sigma}_k^2$ since this uncertainty has only minor impact on the mean squared estimation error of the claims reserve, see. e.g. Mack (1993); Merz and Wüthrich (2008). However, the uncertainty about volatility parameters has a crucial impact on the value-at-risk of the corresponding predictive distribution according to the solvency capital requirement.

To model the predictive distribution the existing literature recommends ei-

ther the Bayesian approach or bootstrapping, see e.g. Björkwall (2011); England and Verrall (1999, 2006); Gisler (2006); Pinheiro et al. (2003). For the one-year risk we refer to Diers and Kraus (2010). In the actuarial practice, bootstrapping is commonly used, since it is easy to implement. However, none of these articles addresses Question 1.1.

This article is a first contribution how to model the solvency capital requirement for the reserve risk with respect to the required confidence level of 99.5% in the sense of Question 1.1. To understand the economic relevance of parameter uncertainty with respect to the unknown future losses, it is important to distinguish between the following two perspectives:

1. the theoretical perspective: From the theoretical perspective both the historical data and the parameter estimate $\hat{\theta}$ are random. The true parameter vector is not known, but fixed.
2. the undertaking's perspective: From this perspective, the historical data and, using a fixed estimation method, the estimate $\hat{\theta}$ are known. There is uncertainty about the true parameter vector θ . From this perspective, the true parameter vector θ appears to be random.

In Section 3 we explain why the actual economic risk relevant for Solvency II is given by the undertaking's perspective reflecting the real situation of the undertaking. For illustration, we use a simple example demonstrating the difference between the theoretical perspective and the undertaking's perspective. For this example we show that the two perspectives lead to different parameter distributions and prove that indeed the parameter distribution corresponding to the undertaking's perspective yields an exact solution to Question 1.1. Moreover, we explain why standard methods of classical reserving based on the theoretical perspective are in general not appropriate to model the impact of parameter uncertainty upon the actual risk of economic losses.

A possibility to model parameter uncertainty from the undertaking's perspective would be the application of the Bayesian approach. Note that these techniques use additional a-priori-information resp. expert judgement (cf. e.g. Wüthrich and Merz (2008) Chapter 4, Verrall (1990) or Peters et al. (2016)).

However, in order to find a solution to Question 1.1 for the reserve risk

we follow a different approach introduced in Fröhlich and Weng (2015) (in the sequel this method is called “inversion method”). In Fröhlich and Weng (2015) the inversion method has been proven to be appropriate to model parameter uncertainty in the sense of Question 1.1 for a wide class of distributions. Insofar, we concentrate on modelling a predictive distribution from the undertaking’s perspective without using any a-priori-information or expert knowledge. In particular, using the inversion method we avoid a sophisticated Markov Chain Monte Carlo simulation, usually necessary to perform a Bayesian analysis.

Referring to Question 1.1 in Section 4 we assess several methods for modelling the risk capital requirement by investigating the probability of solvency (cf. Section 2.3). To illustrate the effects of parameter uncertainty we consider a very simple model - the normal model (cf. England and Verrall (2006); Gisler (2006)). We discuss the bootstrapping approach and present experimental results demonstrating that even for this simple model bootstrapping is not appropriate in the sense of Question 1.1, since it does not guarantee the required solvency level of 99.5% under the consideration of the randomness of the historical data.

In Section 4.3 we adjust the method proposed in Fröhlich and Weng (2015) to derive a risk capital model for the reserve risk achieving the required probability of solvency in good approximation.

2 Parameter uncertainty and reserve risk

2.1 Notation

Random variables are printed in **bold**.

Let $\boldsymbol{\xi}$ be a random variable uniformly distributed on $[0; 1]$. By ξ we denote a fixed realizations of $\boldsymbol{\xi}$.

Let $I \subseteq \mathbb{R}^d$ be a set of parameters and let $\{\mathbf{X}(\theta) | \theta \in I \subseteq \mathbb{R}^d\}$ and $\{F_{\mathbf{X}(\theta)} | \theta \in I \subseteq \mathbb{R}^d\}$ be the corresponding set of random variables resp. the set of corresponding cumulative distribution functions. We use the convenient notation introduced in Fröhlich and Weng (2015): Assume that the cumulative distribution function $F_{\mathbf{X}(\theta)}$ of $\mathbf{X}(\theta)$ is continuous and its inverse exists. We define the map $X : [0; 1] \times I \rightarrow \mathbb{R}$ by $X(\xi, \theta) := F_{\mathbf{X}(\theta)}^{-1}(\xi)$ and use $X(\boldsymbol{\xi}, \theta)$ to denote the random variable $\mathbf{X}(\theta)$.

Throughout the article, $t = 0$ denotes the time corresponding to the current solvency balance sheet and $t = 1$ denotes the end of the one-year period.

2.2 Basic definitions and parameter risk

Let $\mathbf{C}_{i,k}$ denote the cumulative claims payments of accident year i , $0 \leq i \leq n$, up to development year k , $0 \leq k \leq n$. We interpret $\mathbf{C}_{i,k}$ as a random variable for which we observe realizations $C_{i,k}$ for $i + k \leq n$. In the sequel D denotes the observed claims development triangle $\{C_{i,k} : i + k \leq n\}$ which is considered as a realization of a random vector \mathbf{D} . For the sake of simplicity, we neglect the effect from interest rates upon the best estimate reserve.

Using an appropriate reserving method to estimate the ultimate claims payment $\hat{C}_{i,n}$ of accident year $i = 1, \dots, n$, the best estimate reserve is given by $\hat{R}_0^i = \hat{C}_{i,n} - C_{i,n-i}$ for all accident years $i = 1, \dots, n$. For simplicity we assume that all claims are settled after n development years.

The total best estimate reserve is given by $\hat{R}_0 = \sum_{i=1}^n \hat{R}_0^i$. We use the notation $\hat{R}_0(D)$ to emphasize the dependency of \hat{R}_0 on the realization D of \mathbf{D} .

Consider the random payments of the next calendar year

$$\mathbf{Z}_{i,n-i+1} = \mathbf{C}_{i,n-i+1} - C_{i,n-i} \quad (1)$$

for $1 \leq i \leq n$ and set $\mathbf{Z} = \sum_{i=1}^n \mathbf{Z}_{i,n-i+1}$.

As in Diers and Kraus (2010); Merz and Wüthrich (2008) we assume that the best estimate reserve for the same accident years at the end of the one-year period, i.e. at $t = 1$, is determined by the claims observed up to time $t = 1$ (i.e. the claims development triangle D in $t = 0$ extended by the diagonal representing the payments of the next calendar year, which is obtained from the vector of random variables $\vec{\mathbf{Z}} = (\mathbf{Z}_{i,n-i+1} : 1 \leq i \leq n)$ using the chain ladder method. We write $\hat{\mathbf{R}}_1 = \hat{R}_1(D, \vec{\mathbf{Z}})$ to stress this deterministic dependency.

The one-year claims development loss $\mathbf{X} = \mathbf{Z} + \hat{\mathbf{R}}_1 - \hat{R}_0(D)$ describes the possible loss caused by the difference between the best estimate reserve in $t = 0$ and the sum of the expenses for the claims payments within the next year and the expenditure for setting up the reserve at the end of the next year. For simplicity we use \mathbf{S} to denote $\mathbf{Z} + \hat{\mathbf{R}}_1$.

We assume that the distribution of both $\mathbf{S}(\theta)$ and $\mathbf{D}(\theta)$ are known except for the unknown parameter vector θ .

In this paper we use the stochastic chain ladder model introduced by Mack (see Mack (1993, 1999)): Let

$$F_{i,k} = \frac{C_{i,k}}{C_{i,k-1}}.$$

and assume that there exist factors $f_1, \dots, f_n > 0$ and variance parameters $\sigma_1^2, \dots, \sigma_n^2$ such that for all $0 \leq i \leq n$ and $1 \leq k \leq n$ we have

- $E[\mathbf{F}_{i,k} | C_{i,0}, \dots, C_{i,k-1}] = f_k$,
- $\text{Var}[\mathbf{F}_{i,k} | C_{i,0}, \dots, C_{i,k-1}] = \frac{\sigma_k^2}{C_{i,k-1}^\gamma}$ for $\gamma = 0$ or $\gamma = 1$ and
- independence of the accident years: the vectors $(\mathbf{C}_{i,0}, \dots, \mathbf{C}_{i,n})$, $0 \leq i \leq n$, are independent.

The true parameter vector $\theta = (f_1, \sigma_1^2, \dots, f_n, \sigma_n^2)$ is unknown. Unbiased estimates are given by

$$\begin{aligned} \hat{f}_k &= \sum_{i=0}^{n-k} C_{i,k-1}^\gamma F_{i,k} / \sum_{i=0}^{n-k} C_{i,k-1}^\gamma, \\ \hat{\sigma}_k^2 &= \frac{1}{n-k} \sum_{i=0}^{n-k} C_{i,k-1}^\gamma \left(F_{i,k} - \hat{f}_k \right)^2 \end{aligned} \quad (2)$$

for $k = 1, \dots, n-1$. Recall that we assume that all claims are settled after n years. Hence, $\hat{f}_n = 1$ and $\hat{\sigma}_n^2 = 0$. The estimated parameter vector is then given by $\hat{\theta} = (\hat{f}_1, \hat{\sigma}_1^2, \dots, \hat{f}_{n-1}, \hat{\sigma}_{n-1}^2)$.

2.3 Modelled risk and probability of solvency

We define the modelled risk capital and the probability of solvency taking parameter uncertainty into account.

Let \mathbf{X} be a random variable describing a subrisk of the undertaking. For the reserve risk consider the one-year claims development loss

$$\mathbf{X} = \mathbf{S}(\theta) - \hat{R}_0(D)$$

where $\mathbf{S}(\theta) = \mathbf{Z}(\theta) + \hat{R}_1(D, \vec{\mathbf{Z}}(\theta))$ (cf. Section 2.2).

If the parameter vector θ was known, the required risk capital for the one-year reserve risk for the confidence level α would just be the α -quantile of the random variable $\mathbf{X} = \mathbf{X}(\theta)$.

But since the undertaking does not know the parameter vector θ , it does not know the true distribution of \mathbf{X} . Hence, we assume that it can only calculate the risk capital requirement based on the observed historical data D , which is a realization of the random vector \mathbf{D} . The randomness of \mathbf{D} implies that the risk capital requirement **SCR** is also random.

Recalling Question 1.1 we ask for a method to model the standalone risk capital requirement **SCR** for the reserve risk, such that the possible claims development loss \mathbf{X} does not exceed the **SCR** with a probability of 99.5%, i.e.

$$P(\mathbf{X} \leq \mathbf{SCR}) = 99.5\% \quad (3)$$

taking into account that both, \mathbf{X} and **SCR**, are random variables.

Given the observed data $C_{i,k}$ for $i + k \leq n$ we assume that the undertaking models its risk as a predictive distribution by the following two-step procedure:

1. Given a method M and the triangle $D = \{C_{i,k} : i + k \leq n\}$ generate a probability distribution $\mathcal{P} = \mathcal{P}(D; M)$ for the parameter vector $\boldsymbol{\theta}^{sim}$.
2. Define $\mathbf{S}^{model} = \mathbf{S}(\boldsymbol{\theta}^{sim}) := S(\boldsymbol{\xi}, \boldsymbol{\theta}^{sim})$ and consider the modelled claims development loss $\mathbf{X}^{model} = \mathbf{X}(\boldsymbol{\theta}^{sim}) := X(\boldsymbol{\xi}, \boldsymbol{\theta}^{sim})$ (cf. the notation introduced in Subsection 2.1 and 2.2).

The modelled risk $\mathbf{X}^{model} := \mathbf{X}^{model}(\boldsymbol{\theta}^{sim})$ depends on the data D , but also on the method M resp. the chosen parameter distribution \mathcal{P} . Using Monte-Carlo we simulate the mixed distribution of \mathbf{X}^{model} by the two step procedure described above.

Definition 2.1. Let \mathbf{X}^{model} be the modelled risk described above. We call

$$\text{SCR}(\alpha; D; M) := F_{\mathbf{X}^{model}}^{-1}(\alpha)$$

the **modelled risk capital for confidence level α** and

$$P(\mathbf{X} \leq \text{SCR}(\alpha; \mathbf{D}; M)) = P\left(\mathbf{S}(\theta) - \hat{R}_0(\mathbf{D}) \leq \text{SCR}(\alpha; \mathbf{D}; M)\right) \quad (4)$$

the **probability of solvency** according to the corresponding risk capital model subject to the method M .

Note that in (4) not only $\mathbf{S}(\theta)$, but also \mathbf{D} and therefore $\text{SCR}(\alpha; \mathbf{D}; M)$ are considered to be random.

In Section 4 we discuss several methods to determine $\mathbf{X}^{model}(D)$ and assess these methods referring to Question 1.1 by comparing the obtained probability of solvency with the required confidence level.

Remark 2.2. Note that there is a close relation of the probability of solvency to backtesting (cf. Gerrard and Tsanakas (2011), p. 731 or Fröhlich and Weng (2015), Remark 2).

Remark 2.3. Since $\text{SCR}(\alpha; D; M) = F_{\mathbf{S}^{model}}^{-1}(\alpha) - \hat{R}_0(D)$ and $\mathbf{X} = \mathbf{S} - \hat{R}_0(D)$, the solvency requirement $\mathbf{X} \leq \text{SCR}(\alpha; \mathbf{D}; M)$ is equivalent to $\mathbf{S} \leq F_{\mathbf{S}^{model}}^{-1}(\alpha)$, i.e. the best estimate reserve $\hat{R}_0(D)$ cancels out on both sides of the inequality. This shows that the problem of risk capital calculation for the reserve risk in the context of Solvency II according to Question 1.1 differs considerably from the objective of classical reserving methods focussing on the estimation error according to the best estimate reserve $\hat{R}_0(D)$.

3 Parameter uncertainty from the undertaking's perspective

Recall the definition of the terms “theoretical perspective” and “undertaking's perspective” given in the introduction.

The objective of this section is to provide an intuitive understanding of the impact of parameter uncertainty for the reserve risk from an economic point of view. For this purpose it is crucial to recognize that the actual situation of the undertaking can be characterized by the following simple, but fundamental observations:

- i) The observed development triangle D is fixed. Hence, for a given estimation method both the parameter estimate $\hat{\theta}$ and the best estimate reserve $\hat{R}_0(D)$ are also fixed. In particular, there is uncertainty about θ - not about $\hat{\theta}$.
- ii) The real economic risk results from the true distribution of future claims depending on the unknown true parameter vector θ . The parameter risk

arises from the uncertainty about θ . However, the parameter estimation does not (directly) influence the true distribution of future claims.

These observations make obvious that the actual economic risk of the undertaking relevant for Solvency II is given by the undertaking's perspective. In particular, in order to model the real economic reserve risk it does not make sense to use a predictive distribution of future claims payments resp. of \mathbf{S}^{model} directly based on the distribution of the estimate $\hat{\theta}$. We conclude that the theoretical perspective is not appropriate to model the impact of parameter uncertainty from the economic point of view of the undertaking. More precisely:

A predictive distribution modelling future losses \mathbf{X}^{model} based on the parameter distribution of the estimate $\hat{\theta}$ is, in general, not appropriate to represent the impact of parameter uncertainty upon the real risk of the undertaking, which arises from the uncertainty about the true parameter θ corresponding to the actual economic losses $\mathbf{X} = \mathbf{X}(\theta)$.

Summarizing the discussion above, from the economic point of view of the undertaking parameter risk is defined as follows:

Definition 3.1. The **parameter risk** from the undertaking's perspective refers to the uncertainty about the true parameter vector θ corresponding to the random variable \mathbf{S} conditioned on the fixed observed triangle D .

Before considering the rather complex reserve risk we illustrate the difference between the theoretical and the undertaking's perspective using the simple example of the normal distribution $N(0; \sigma^2)$ with fixed, but unknown parameter σ^2 .

Example 3.2. Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. with $\mathbf{X}_i \sim \mathbf{X} = \sigma \cdot \mathbf{Z}$, $\mathbf{Z} \sim N(0; 1)$ for $i = 1, \dots, n$. An unbiased estimate of the parameter σ^2 is given by

$$\hat{\sigma}^2 := \frac{1}{n} \cdot \sum_i \mathbf{X}_i^2 = \frac{\sigma^2}{n} \cdot \sum_i \mathbf{Z}_i^2 = \frac{\sigma^2}{n} \cdot \mathbf{M} \quad (5)$$

where \mathbf{Z}_i are i.i.d. with $\mathbf{Z}_i \sim N(0; 1)$ and $\mathbf{M} := \sum \mathbf{Z}_i^2$ is $\chi^2(n)$ distributed with n degrees of freedom.

1. From the *theoretical perspective* $\hat{\sigma}^2$ has the distribution

$$\hat{\sigma}^2 = \frac{\sigma^2}{n} \cdot \mathbf{M} \sim \frac{\sigma^2}{n} \cdot \chi^2(n) \quad (A),$$

where $\chi^2(n)$ is the χ^2 -distribution with n degrees of freedom.

2. From the *undertaking's perspective* the estimate $\hat{\sigma}^2$ is given, but there is uncertainty about the true parameter σ^2 . From this perspective the uncertainty about σ^2 is due to the fact that the undertaking does not know the realization of the random factor \mathbf{M} . If the undertaking knew the realization M of \mathbf{M} , it would be able to conclude the value of the true parameter

$$\sigma^2 = n \cdot \hat{\sigma}^2 / M$$

by solving Equation (A) for σ^2 .

In this sense the uncertainty of the undertaking about σ^2 is equivalent to the uncertainty about the realization of \mathbf{M} . Thus, based on the fixed observed parameter estimate $\hat{\sigma}^2$ the uncertainty of the undertaking with respect to σ^2 can be expressed by

$$\sigma_{sim}^2 = n \cdot \hat{\sigma}^2 / \mathbf{M}' \sim n \cdot \hat{\sigma}^2 / \chi^2(n) \quad (B)$$

where the random variable $\mathbf{M}' \sim \chi^2(n)$ representing the undertaking's uncertainty about the realization of \mathbf{M} is independent of \mathbf{M} , since the undertaking does not know the realization M of \mathbf{M} .

Let us explain the distributional assumptions in more detail: Note that the distribution of \mathbf{M} is independent of σ^2 and, without any knowledge of $\sigma^2 \in \mathbb{R}^+$, the sole observation of $\hat{\sigma}^2$ does not reveal any additional information about the ratio $\hat{\sigma}^2 / \sigma^2$, resp. the realization M of \mathbf{M} , from the undertaking's perspective. This justifies the assumption that the distribution of \mathbf{M}' is both independent of the observation $\hat{\sigma}^2$ and still equal to the original $\chi^2(n)$ -distribution of \mathbf{M} .

Indeed, the parameter distribution (B) yields an exact solution to Question 1.1 for this simple example: Let $\mathbf{X}^{model}(\hat{\sigma}) = \sigma_{sim} \cdot \mathbf{Z}' = \hat{\sigma} \cdot \sqrt{\frac{n}{\mathbf{M}'}} \cdot \mathbf{Z}'$ for some standard normally distributed random variable \mathbf{Z}' independent of \mathbf{Z} , \mathbf{M} and \mathbf{M}' . Note that \mathbf{X}^{model} can be written as $\hat{\sigma} \cdot \mathbf{T}'$ where $\mathbf{T}' = \sqrt{n/\mathbf{M}'} \cdot \mathbf{Z}'$ is t -distributed with n degrees of freedom and denote the α -quantile of \mathbf{T}' by $F_{\mathbf{T}'}^{-1}(\alpha)$. For

$SCR(\hat{\sigma}) := F_{\mathbf{X}^{model}(\hat{\sigma})}^{-1}(\alpha) = \hat{\sigma} \cdot F_{\mathbf{T}'}^{-1}(\alpha)$ we derive

$$\begin{aligned} P(\mathbf{X} \leq SCR(\hat{\sigma})) &= P(\sigma \cdot \mathbf{Z} \leq \hat{\sigma} \cdot F_{\mathbf{T}'}^{-1}(\alpha)) \\ &= P\left(\sigma \cdot \mathbf{Z} \leq \sigma \cdot \sqrt{\frac{\mathbf{M}}{n}} \cdot F_{\mathbf{T}'}^{-1}(\alpha)\right) \\ &= P\left(\mathbf{Z} \cdot \sqrt{\frac{n}{\mathbf{M}}} \leq F_{\mathbf{T}'}^{-1}(\alpha)\right) = \alpha. \end{aligned}$$

This proves that the modelled risk capital $SCR(\hat{\sigma}) := F_{\mathbf{X}^{model}(\hat{\sigma})}^{-1}(\alpha)$ attains the required probability of solvency, i.e. setting $\alpha = 99.5\%$ we solved Question 1.1 for this example.

The distributions (A) and (B) do not coincide. Note that the density function of the $\chi^2(n)$ -distribution corresponding to (A) is equal to

$$f(x) = const \cdot x^{\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right),$$

i.e. it decreases exponentially, whereas the distribution $1/\chi^2(n)$ corresponding to (B) has the density function

$$\frac{1}{x^2} \cdot f\left(\frac{1}{x}\right) = const \cdot x^{-\frac{n}{2}-1} \cdot \exp\left(-\frac{1}{2x}\right).$$

Hence, distribution (B) representing the undertaking's perspective has an heavy tail (no exponential decay) in contrast to the distribution (A).

The example demonstrates that the two perspectives are not equivalent. There is no symmetry or equivalence between the uncertainty about $\hat{\theta}$ from the theoretical perspective and the uncertainty about θ from the undertaking's perspective. In particular, in this example the parameter risk from the economical relevant perspective from the undertaking would be significantly underestimated using the parameter distribution (A) corresponding to the theoretical perspective.

Remark 3.3. The considerations in Example 3.2 can be generalized to a wider class of distribution families. For a general description of the inversion method as well as for the general statistical questions like its relation to fiducial inference and the frequentist approach in the context of confidence intervals, we refer to Fröhlich and Weng (2015).

In this contribution we restrict to the undertaking's perspective which is based on the observed claims development triangle D . Note that the extensive literature about reserving concerning the mean squared error of estimation as well as the contributions concerning the bootstrapping approach always adopt the theoretical perspective. Thus, these approaches are not appropriate to model the impact of parameter uncertainty upon the actual risk of economic losses from the undertaking's perspective.

Indeed, in Section 4.2 we demonstrate that bootstrapping does not yield a solution to Question 1.1.

4 Appropriateness of the methods for the normal model

Referring to Question 1.1 in this section we assess several methods for calculating the risk capital for the one-year reserve risk by comparing the probability of solvency attained by the respective method to the required confidence level. To avoid technical complications we concentrate on a rather simple model - the normal model England and Verrall (2006); Gisler (2006) based on Mack's chain ladder model (cf. Subsection 2.2).

We assume that the individual chain ladder factors $\mathbf{F}_{i,k} = \mathbf{C}_{i,k}/C_{i,k-1}$ conditioned on $\{C_{i,0}, \dots, C_{i,k-1}\}$ are normally distributed, i.e. there exists parameters $f_k > 0$ and σ_k independent of the specific accident year i such that

$$\mathbf{F}_{i,k} | \{C_{i,0}, \dots, C_{i,k-1}\} \sim N \left(f_k; \frac{\sigma_k^2}{C_{i,k-1}^\gamma} \right)$$

for $0 \leq i \leq n$ and $1 \leq k \leq n$ and $\gamma = 0$ or $\gamma = 1$.

Thus, given $C_{i,k-1}$ we can write

$$\mathbf{F}_{i,k} = f_k + \frac{\sigma_k}{\sqrt{C_{i,k-1}^\gamma}} \cdot \zeta_{i,k} \quad (6)$$

with standard normally distributed residues $\zeta_{i,k}$ for $0 \leq i \leq n$ and $1 \leq k \leq n$. We denote the set of realized ("true", but still unknown) residues by $\mathcal{R} = \{\zeta_{i,k} : i+k \leq n\}$. Moreover, we assume that $C_{i,0}$ is normally distributed with mean f_0 and variance σ_0^2 for $0 \leq i \leq n$. Note that the realizations of $C_{i,0}$ have already been observed. Hence, for the modelling of the reserve risk an

estimation of the parameters f_0 and σ_0^2 is not necessary and the parameter uncertainty with respect to these parameters is not relevant.

Unbiased estimates \hat{f}_k resp. $\hat{\sigma}_k^2$ of f_k and σ_k^2 for $1 \leq k \leq n-1$ where $\gamma = 0$ or $\gamma = 1$ are given by (2). We assume that all payments are settled after n development years, that is, $f_n = 1$ and $\sigma_n^2 = 0$, and we set $\hat{f}_n = 1$ and $\hat{\sigma}_n^2 = 0$. Note that the payments of the next calendar year are given by $\mathbf{Z}(\theta) = \sum_{i=1}^n \mathbf{Z}_{i,n-i+1}(\theta_{n-i+1})$ with

$$\mathbf{Z}_{i,n-i+1}(\theta_{n-i+1}) = \mathbf{C}_{i,n-i+1}(\theta_{n-i+1}) - C_{i,n-i}, \quad i = 1 \dots, n,$$

for $\theta = (\theta_1, \dots, \theta_{n-1})$ with $\theta_{n-i+1} = (f_{n-i+1}, \sigma_{n-i+1}^2)$ where

$$\mathbf{C}_{i,n-i+1}(\theta_{n-i+1}) = C_{i,n-i} \cdot \mathbf{F}_{i,n-i+1}(\theta_{n-i+1})$$

and $\mathbf{F}_{i,n-i+1} = f_{n-i+1} + \frac{\sigma_{n-i+1}}{\sqrt{C_{i,n-i}^\gamma}} \cdot \zeta_{i,n-i+1}$ for independent, standard normally distributed residues $\zeta_{i,n-i+1}$.

We assume that the parameter vector $\theta = (f_1, \sigma_1^2, f_2, \sigma_2^2, \dots, f_{n-1}, \sigma_{n-1}^2)$ is unknown.

Using the notation introduced in Subsection 2.2 we model the incremental payment $\mathbf{Z}_{i,n-i+1}$ by $\mathbf{Z}_{i,n-i+1}^{model} = \mathbf{C}_{i,n-i+1}^{model} - C_{i,n-i}$ where $\mathbf{C}_{i,n-i+1}^{model} = C_{i,n-i} \cdot \mathbf{F}_{i,n-i+1}^{sim}$ and

$$\mathbf{F}_{i,n-i+1}^{sim} = f_{n-i+1}^{sim} + \frac{\sigma_{n-i+1}^{sim}}{\sqrt{C_{i,n-i}^\gamma}} \cdot \zeta'_{i,n-i+1}$$

for a random parameter vector $\theta_{n-i+1}^{sim} = (f_{n-i+1}^{sim}, (\sigma_{n-i+1}^{sim})^2)$ determined by some method M and independent of $\zeta'_{i,n-i+1}$ with $\zeta'_{i,n-i+1} \sim N(0; 1)$ independent of $\zeta_{i,n-i+1}$.

We model the reserve $\hat{R}_1(D, \vec{\mathbf{Z}}^{model}) = \hat{R}_1(D, (\mathbf{Z}_{i,n-i+1}^{model} : 1 \leq i \leq n))$ using the chain ladder method with either $\gamma = 0$ or $\gamma = 1$.

Recalling the notation introduced in Subsection 2.2 and Subsection 2.3 the “true” claims development loss is given by

$$\mathbf{X} = \mathbf{Z}(\theta) + \hat{R}_1(D; \vec{\mathbf{Z}}(\theta)) - \hat{R}_0(D)$$

and the modelled claims development loss is equal to

$$\mathbf{X}^{model} = \mathbf{Z}^{model} + \hat{R}_1(D, \vec{\mathbf{Z}}^{model}) - \hat{R}_0(D)$$

where $\vec{\mathbf{Z}} = (\mathbf{Z}_{i,n-i+1} : 1 \leq i \leq n)$ and $\mathbf{Z}^{model} = \sum_{i=1}^n \mathbf{Z}_{i,n-i+1}^{model}$.

4.1 Without modelling parameter risk

In this section we consider the approach without modelling parameter risk, i.e. we set $\mathbf{f}_k^{sim} \equiv \hat{f}_k$ and $\boldsymbol{\sigma}_k^{sim} \equiv \hat{\sigma}_k$ (cf. Subsection 2.3). Note that the common approach used in practice is bootstrapping which will be considered in Subsection 4.2.

We model the cumulative claims for the next business year by $\mathbf{C}_{i,n-i+1}^{model,without} = C_{i,n-i} \cdot \mathbf{F}_{i,n-i+1}^{sim,without}$ with

$$\mathbf{F}_{i,n-i+1}^{sim,without} = \hat{f}_{n-i+1} + \frac{\hat{\sigma}_{n-i+1}}{\sqrt{C_{i,n-i}^\gamma}} \cdot \boldsymbol{\zeta}_{i,n-i+1}$$

for $\boldsymbol{\zeta}_{i,n-i+1}$ independent, normally distributed residues.

We set

$$\mathbf{Z}_{without}^{model}(D) = \sum_{i=1}^n \mathbf{Z}_{i,n-i+1}^{model,without} = \sum_{i=1}^n \left(\mathbf{C}_{i,n-i+1}^{model,without} - C_{i,n-i} \right)$$

and obtain the reserve $\hat{R}_1(D, \vec{\mathbf{Z}}_{without}^{model}) = \hat{R}_1(D, (\mathbf{Z}_{i,n-i+1}^{model,without} : 1 \leq i \leq n))$. The risk capital $\text{SCR}(\alpha; D; without)$ is defined as the α -quantile of

$$\mathbf{X}_{without}^{model}(D) = \mathbf{Z}_{without}^{model}(D) + \hat{R}_1(D, \vec{\mathbf{Z}}_{without}^{model}) - \hat{R}_0(D).$$

The estimates \hat{f}_k and $\hat{\sigma}_k^2$ depend on the realization D of the random claims development triangle \mathbf{D} .

Using the assumptions given in Subsection B.1 and following the general approach described in Subsection B.2 we derive the results for the probability of solvency presented in Table 1.

α	$\gamma = 0$	$\gamma = 1$
90%	84.98%	85.80%
95%	91.19%	91.35%
99%	96.97%	97.07%
99.5%	97.97%	98.09%

Table 1: Solvency probabilities $P(\mathbf{X} \leq \text{SCR}(\alpha; \mathbf{D}; without))$ for the approach without the consideration of parameter risk, for different quantiles and for $\gamma = 0$ resp. $\gamma = 1$

Conclusion 4.1. Neglecting parameter uncertainty leads to a probability of solvency which is significantly lower than the required confidence level.

4.2 Bootstrapping

In this section we consider the popular approach using bootstrapping. There are numerous variants of the bootstrapping approach; we follow Subsection 7.4 in Wüthrich and Merz (2008).

In the sequel we describe how to determine the modelled risk $\mathbf{X}_{model}^{BT}(D)$ using bootstrapping: Again, we set \hat{f}_k and $\hat{\sigma}_k^2$ as in (2). Given \hat{f}_k and $\hat{\sigma}_k^2$ we estimate the residues by

$$\hat{\zeta}_{i,k} = \left(\frac{F_{i,k} - \hat{f}_k}{\hat{\sigma}_k} \right) \cdot \sqrt{C_{i,k-1}^\gamma} \text{ for } i + k \leq n, k < n$$

and consider the set $\hat{\mathcal{R}} := \{\hat{\zeta}_{i,k} : i + k \leq n, k < n\}$.

As pointed out in Wüthrich and Merz (2008), Section 7.4, Equation 7.23, the variance of the residues $\hat{\zeta}_{i,k} \in \hat{\mathcal{R}}$ is smaller than 1. More precisely,

$$\text{Var}(\hat{\zeta}_{i,k} | C_{0,k-1}, \dots, C_{n-k,k-1}) = 1 - \frac{C_{i,k-1}^\gamma}{\sum_{i=0}^{n-k} C_{i,k-1}^\gamma} < 1.$$

We adjust the residues accordingly (cf. Equation 7.24 in Wüthrich and Merz (2008)) and obtain the set \mathcal{R}^* .

We follow the conditional approach in Wüthrich and Merz (2008), Section 7.4.2 to construct a bootstrapping distribution of $\boldsymbol{\theta}^{sim}$ by Monte-Carlo simulation.

A scenario of the bootstrapping distribution $(\mathbf{f}_k^{sim,BT}, (\boldsymbol{\sigma}_k^{sim,BT})^2)$ is constructed as follows: We determine the chain ladder factors $\mathbf{f}_k^{sim,BT}$ by

$$\mathbf{f}_k^{sim,BT} = \sum_{i=0}^{n-k} \frac{C_{i,k-1}^\gamma}{\sum_{h=0}^{n-k} C_{h,k-1}^\gamma} \cdot \mathbf{F}_{i,k}^{*,BT}$$

with

$$\mathbf{F}_{i,k}^{*,BT} = \hat{f}_k + \frac{\hat{\sigma}_k}{\sqrt{C_{i,k-1}^\gamma}} \cdot \boldsymbol{\zeta}_{i,k}^*$$

where $\zeta_{i,k}^*$ is chosen randomly from \mathcal{R}^* and set

$$\left(\sigma_k^{sim,BT}\right)^2 = \frac{1}{n-k} \cdot \sum_{i=0}^{n-k} C_{i,k-1}^\gamma \cdot (\mathbf{F}_{i,k}^{*,BT} - \mathbf{f}_k^{sim,BT})^2.$$

We then define

$$\mathbf{F}_{1,n}^{sim,BT} \equiv 1 \text{ and } \mathbf{F}_{i,n-i+1}^{sim,BT} = \mathbf{f}_{n-i+1}^{sim,BT} + \frac{\sigma_{n-i+1}^{sim,BT}}{\sqrt{C_{i,n-i}^\gamma}} \cdot \zeta'_{i,n-i+1} \text{ for } i = 2, \dots, n$$

where $\zeta'_{i,n-i+1} \sim N(0;1)$ are i.i.d. random variables independent of the bootstrapped parameters. This defines $\mathbf{C}_{i,n-i+1}^{model,BT} = \mathbf{F}_{i,n-i+1}^{sim,BT} \cdot C_{i,n-i}$ and

$$\mathbf{Z}_{BT}^{model}(D) = \sum_{i=1}^n \mathbf{Z}_{i,n-i+1}^{model,BT} = \sum_{i=1}^n \left(\mathbf{C}_{i,n-i+1}^{model,BT} - C_{i,n-i} \right).$$

The risk capital $\text{SCR}(\alpha; D; BT)$ is defined as the α -quantile of

$$\mathbf{X}_{BT}^{model}(D) = \mathbf{Z}_{BT}^{model}(D) + \hat{R}_1(D, \vec{\mathbf{Z}}_{BT}^{model}) - \hat{R}_0(D)$$

where $\vec{\mathbf{Z}}_{BT}^{model} = (\mathbf{Z}_{i,n-i+1}^{model,BT} : 1 \leq i \leq n)$.

Using the assumptions given in Subsection B.1 and following the general approach described in Appendix B.2 we derive the results for the probability of solvency presented in Table 2.

α	$\gamma = 0$	$\gamma = 1$
90%	88.57%	89.05%
95%	93.68%	94.02%
99%	98.27%	98.63%
99.5%	99.08%	99.15%

Table 2: Solvency probabilities $P(\mathbf{X} \leq \text{SCR}(\alpha; \mathbf{D}; BT))$ for the bootstrapping approach, for different quantiles and for $\gamma = 0$ and $\gamma = 1$

Conclusion 4.2. The bootstrapping approach does not attain the required confidence level.

4.3 The inversion method for the normal model

The “inversion method” introduced in Fröhlich and Weng (2015) expresses the parameter estimate $\hat{\theta}$ in terms of the true parameter θ and to invert this relation to obtain an expression of θ in terms of $\hat{\theta}$.

Since we assume that all payments are settled after n development years, we set $\mathbf{f}_n^{sim} \equiv 1$ and $\boldsymbol{\sigma}_n^{sim} \equiv 0$ and apply the idea of the inversion method to the parameter vector $\theta = (f_1, \sigma_1^2, f_2, \sigma_2^2, \dots, f_{n-1}, \sigma_{n-1}^2)$ of the normal model: Inserting

$$\mathbf{F}_{i,k} = f_k + \frac{\sigma_k}{\sqrt{C_{i,k-1}^\gamma}} \cdot \boldsymbol{\zeta}_{i,k}$$

with $\boldsymbol{\zeta}_{i,k}$ standard normally distributed into (2) yields for $k = 1, \dots, n-1$

$$\begin{aligned} \hat{\mathbf{f}}_k &= \sum_{i=0}^{n-k} \mathbf{F}_{i,k} \cdot \frac{C_{i,k-1}^\gamma}{\sum_{h=0}^{n-k} C_{h,k-1}^\gamma} \\ &= f_k + \sigma_k \cdot \sum_{i=0}^{n-k} \frac{\boldsymbol{\zeta}_{i,k}}{\sqrt{C_{i,k-1}^\gamma}} \cdot \frac{C_{i,k-1}^\gamma}{\sum_{h=0}^{n-k} C_{h,k-1}^\gamma} = f_k + \sigma_k \cdot \mathbf{R}_k \text{ and} \\ \hat{\sigma}_k^2 &= \frac{1}{n-k} \sum_{i=0}^{n-k} C_{i,k-1}^\gamma (\mathbf{F}_{i,k} - \hat{\mathbf{f}}_k)^2 \\ &= \frac{1}{n-k} \cdot \sigma_k^2 \cdot \sum_{i=0}^{n-k} C_{i,k-1}^\gamma \left(\frac{\boldsymbol{\zeta}_{i,k}}{\sqrt{C_{i,k-1}^\gamma}} - \sum_{j=0}^{n-k} \frac{\boldsymbol{\zeta}_{j,k}}{\sqrt{C_{j,k-1}^\gamma}} \frac{C_{j,k-1}^\gamma}{\sum_{h=0}^{n-k} C_{h,k-1}^\gamma} \right)^2 \\ &= \sigma_k^2 \cdot \mathbf{M}_k \end{aligned}$$

with

$$\mathbf{R}_k = \sum_{i=0}^{n-k} \boldsymbol{\zeta}_{i,k} \cdot \frac{\sqrt{C_{i,k-1}^\gamma}}{\sum_{h=0}^{n-k} C_{h,k-1}^\gamma} \quad (7)$$

and

$$\begin{aligned} \mathbf{M}_k &= \frac{1}{n-k} \sum_{i=0}^{n-k} C_{i,k-1}^\gamma \left(\frac{\boldsymbol{\zeta}_{i,k}}{\sqrt{C_{i,k-1}^\gamma}} - \sum_{j=0}^{n-k} \frac{\boldsymbol{\zeta}_{j,k}}{\sqrt{C_{j,k-1}^\gamma}} \frac{C_{j,k-1}^\gamma}{\sum_{h=0}^{n-k} C_{h,k-1}^\gamma} \right)^2 \\ &= \sum_{i=0}^{n-k} C_{i,k-1}^\gamma \left(\frac{\boldsymbol{\zeta}_{i,k}}{\sqrt{C_{i,k-1}^\gamma}} - \mathbf{R}_k \right)^2. \end{aligned} \quad (8)$$

Solving these equations for f_k resp. σ_k^2 for $1 \leq k \leq n-1$ defines a probability distribution of the unknown parameter vector (f_k, σ_k^2) given by

$$\mathbf{f}_k^{sim} := \hat{f}_k - \sqrt{\frac{1}{\mathbf{M}'_k}} \cdot \hat{\sigma}_k \cdot \mathbf{R}'_k \quad \text{and} \quad (\sigma_k^{sim})^2 := \frac{1}{\mathbf{M}'_k} \cdot \hat{\sigma}_k^2 \quad (9)$$

where \mathbf{R}'_k resp. \mathbf{M}'_k are the modelled random variables of the same distribution as \mathbf{R}_k and \mathbf{M}_k , but independent of \mathbf{R}_k and \mathbf{M}_k obtained by replacing $\zeta_{i,k}$ in (7) and (8) by independent $\zeta'_{i,k} \sim N(0; 1)$.

To model the claims development loss of the next calendar year we consider (9) for $k = n - i + 1$ and $i = 2, \dots, n$:

$$\mathbf{f}_{n-i+1}^{sim} = \hat{f}_{n-i+1} - \sigma_{n-i+1}^{sim} \mathbf{R}'_{n-i+1} \quad \text{and} \quad (\sigma_{n-i+1}^{sim})^2 = \frac{\hat{\sigma}_{n-i+1}^2}{\mathbf{M}'_{n-i+1}}.$$

Let

$$\mathbf{F}_{i,n-i+1}^{sim} := \mathbf{f}_{n-i+1}^{sim} + \frac{\sigma_{n-i+1}^{sim}}{\sqrt{C_{i,n-i}^\gamma}} \cdot \zeta'_{i,n-i+1} \quad (10)$$

and $\mathbf{Z}_{i,n-i+1}^{model} := (\mathbf{F}_{n-i+1}^{sim} - 1) \cdot C_{i,n-i}$ for $i = 1, \dots, n$.

However, modelling the claims development result directly by $\mathbf{X}^{model} = \sum \mathbf{Z}_{i,n-i+1}^{model} + \hat{R}_1(D, \vec{\mathbf{Z}}^{model}) - \hat{R}_0(D)$ with $\vec{\mathbf{Z}}^{model} = (\mathbf{Z}_{i,n-i+1}^{model} : 1 \leq i \leq n)$ yields a risk capital model which is too conservative, i.e. setting $SCR(\alpha; D) := F_{\mathbf{X}^{model}}^{-1}(\alpha)$ yields $P(\mathbf{X}^{model} \leq SCR(\alpha; D)) > \alpha$ for e.g. $\alpha = 99.5\%$ (see Fröhlich and Weng (2016) for a comprehensive discussion).

Therefore, we need to adjust the inversion method to derive an risk capital model leading to a significantly better approximation of the desired probability of solvency.

For the adjustment of the inversion method we introduce a stochastic correction factor of the same form as suggested in Fröhlich and Weng (2016):

$$\mathbf{a}_{sim} := \left(\sum_{i=2}^n \frac{\hat{w}_{n-i+1}}{\mathbf{M}'_{n-i+1}} \cdot \sum w_{n-i+1} \cdot \mathbf{M}'_{n-i+1} \right)^{-\frac{1}{2}} \quad (11)$$

with weights w_{n-i+1} , $2 \leq i \leq n$, defined by

$$w_{n-i+1} = \frac{(\sigma_{n-i+1})^2 \cdot C_{i,n-i}^2 \cdot \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{j=2}^n (\sigma_{n-j+1})^2 \cdot C_{j,n-j}^2 \cdot \left(\frac{1}{C_{j,n-j}^\gamma} + \frac{1}{\sum_{l=0}^{j-1} C_{l,n-j}^\gamma} \right)}$$

and

$$\hat{w}_{n-i+1} = \frac{(\hat{\sigma}_{n-i+1})^2 \cdot C_{i,n-i}^2 \cdot \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{j=2}^n (\hat{\sigma}_{n-j+1})^2 \cdot C_{j,n-j}^2 \cdot \left(\frac{1}{C_{j,n-j}^\gamma} + \frac{1}{\sum_{l=0}^{j-1} C_{l,n-j}^\gamma} \right)}. \quad (12)$$

Set $\hat{Z}_{i,n-i+1} = (\hat{f}_{n-i+1} - 1) \cdot C_{i,n-i}$ and

$$\mathbf{Z}_{i,n-i+1}^{model,adj} = (1 - \mathbf{a}_{sim}) \hat{Z}_{i,n-i+1} + \mathbf{a}_{sim} \cdot \mathbf{Z}_{i,n-i+1}^{model}$$

for $i = 1, \dots, n$. Note that $\mathbf{Z}_{i,n-i+1}^{model,adj}$ depends on the observed triangle D and the corresponding set of historical residues $\mathcal{R} = \{\zeta_{i,k} : i + k \leq n\}$ (cf. Equation 6). The following theorem holds independently of the particular choice of the parameter vector θ .

Theorem 4.3. *Let $\mathbf{Z}_{adj}^{model} := \sum_{i=1}^n \mathbf{Z}_{i,n-i+1}^{model,adj}$, $\mathcal{R} = \{\zeta_{i,k} : i + k \leq n\}$ be the set of realized residues and let $SCR_{\mathbf{Z}}(\alpha; \mathcal{R}; M)$ be the α -quantile of \mathbf{Z}_{adj}^{model} . Then*

$$P \left(\sum_{i=1}^n \mathbf{Z}_{i,n-i+1} \leq SCR_{\mathbf{Z}}(\alpha; \mathcal{R}; M) \right) = \alpha.$$

Proof. See Appendix A. □

Note that, since σ_{n-i+1} is unknown, we can only estimate \mathbf{a}_{sim} . In the sequel we use the following estimate

$$\hat{\mathbf{a}}_{sim} := \left(\sum_{i=2}^n \frac{\hat{w}_{n-i+1}}{\mathbf{M}'_{n-i+1}} \cdot \sum_{i=2}^n \hat{w}_{n-i+1} \cdot \mathbf{M}'_{n-i+1} \right)^{-\frac{1}{2}}.$$

Theorem 4.3 motivates to set

$$\hat{\mathbf{Z}}_{i,n-i+1}^{model,adj} := (1 - \hat{\mathbf{a}}_{sim}) \cdot \hat{Z}_{i,n-i+1} + \hat{\mathbf{a}}_{sim} \cdot \mathbf{Z}_{i,n-i+1}^{model}$$

and $\hat{\mathbf{Z}}_{adj}^{model} := \sum_{i=1}^n \hat{\mathbf{Z}}_{i,n-i+1}^{model,adj}$.

Define the modelled risk by

$$\hat{\mathbf{X}}_{adj}^{model}(D) = \hat{\mathbf{Z}}_{adj}^{model}(D) + \hat{R}_1(D, (\hat{\mathbf{Z}}_{i,n-i+1}^{model,adj} : 1 \leq i \leq n)) - \hat{R}_0(D)$$

and model the risk capital $SCR(\alpha; D; model, adj)$ as the α -quantile of $\hat{\mathbf{X}}_{adj}^{model,adj}(D)$. Again we consider the example given in Subsection B.1.

α	$\gamma = 0$	$\gamma = 1$
90%	89.92%	89.76%
95%	95.06%	94.89%
99%	99.03%	98.94%
99.5%	99.51%	99.48%

Table 3: Solvency probabilities $P(\mathbf{X} \leq \text{SCR}(\alpha; \mathbf{D}; \text{model}, \text{adj}))$ for the modified inversion method for different quantiles and for $\gamma = 0$ and $\gamma = 1$

Remark 4.4. Note that the probability of solvency defined in (4) depends on the random triangle \mathbf{D} . In Theorem 4.3 we only consider the probability of solvency depending on the randomness of the residues \mathcal{R} for fixed weights $C_{i,k}$. However, the experimental results in Table 3 demonstrate that the method also works for the more general situation of a random \mathbf{D} .

Conclusion 4.5. The risk capital model based on the adjustment of the inversion method using the stochastic correction factor $\hat{\mathbf{a}}_{sim}$ yields a probability of solvency very close to the required confidence levels, i.e. it provides an answer to Question 1.1 posed in the introduction in good approximation.

4.4 Effect on the risk capital

We consider the effect on the risk capital calculation for an explicit example. Consider the claims development triangle taken from Merz and Wüthrich (2008):

	Development year								
	0	1	2	3	4	5	6	7	8
0	2,202,584	3,210,449	3,468,122	3,545,070	3,621,627	3,644,636	3,669,012	3,674,511	3,678,633
1	2,350,650	3,553,023	3,783,846	3,840,067	3,865,187	3,878,744	3,898,281	3,902,425	
2	2,321,885	3,424,190	3,700,876	3,798,198	3,854,755	3,878,993	3,898,825		
3	2,171,487	3,165,274	3,395,841	3,466,453	3,515,703	3,548,422			
4	2,140,328	3,157,079	3,399,262	3,500,520	3,585,812				
5	2,290,664	3,338,197	3,550,332	3,641,036					
6	2,148,216	3,219,775	3,428,335						
7	2,143,728	3,158,581							
8	2,144,738								

The chain ladder reserve $\hat{R}_0(D)$ for $\gamma = 0$ equals to 2,243,574 Euro and for $\gamma = 1$ equals to 2,237,826 Euro.

The risk capital calculation yields the following results for the modelled risk

capital with respect of the 99.5%-quantile using the approaches discussed in the previous sections:

	without parameter uncertainty	with bootstrapping	with adjusted inversion method
$\gamma = 0$	191,589	216,115	227,182
$\gamma = 1$	194,916	216,365	226,980

5 Summary and Outlook

This article deals with the internal modelling of parameter uncertainty for the reserve risk. We pointed out that for the consideration of parameter uncertainty, the undertaking's perspective is the adequate perspective referring to the real risk of economic losses. Therefore, in order to model parameter uncertainty for the reserve risk in the context of Solvency II it is not appropriate to apply methods of classical reserving designed to measure the prediction error from the theoretical perspective.

Considering the probability of solvency already introduced in Gerrard and Tsanakas (2011); Fröhlich and Weng (2015) we assessed several methods to model parameter uncertainty for risk capital calculations considering a very simple model - the normal model. In particular, we demonstrate that the popular bootstrapping approach does not guarantee the required probability of solvency. We then presented an adjustment of the inversion method introduced in Fröhlich and Weng (2015) achieving the required probability of solvency in good approximation.

The main message of our article is not to recommend the usage of the normal model (together with the inversion method). Rather we stress the importance of modelling the solvency capital requirement in such a way that it meets the desired confidence level of 99.5% - even under the consideration of parameter uncertainty. The normal model is just used for illustration.

There are still many questions left for future research:

1. The normal model is very simple and rarely used in practice. For other well-established models the question how to guarantee the required probability of solvency is still open.

2. Does there exist a parameter distribution that guarantees the required probability of solvency simultaneously on every aggregation level (i.e. on the level of every single development factor, every single accident year, every line of business as well as on the level of the overall risk) without using any correction factor when proceeding from one aggregation level to another (cf. Fröhlich and Weng (2016))?
3. Throughout this contribution we assumed that all claims are settled after n years. In particular, we did not address the problem of parameter uncertainty in the context of tail modelling.

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A Proof of Theorem 3.4

For the chain ladder method with $\gamma = 0$ the proof follows directly from Corollary 4.2 in Fröhlich and Weng (2016). For the convenience of the reader, we adjust the proof to include both cases, $\gamma = 0$ and $\gamma = 1$.

Proof. Inserting (12) and (9) into the definition of \mathbf{a}_{sim} (cf. Equation 11) yields

$$\mathbf{a}_{sim} = \sqrt{\frac{\sum_{i=2}^n (\hat{\sigma}_{n-i+1})^2 \cdot C_{i,n-i}^2 \cdot \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n (\boldsymbol{\sigma}_{n-i+1}^{sim})^2 \cdot C_{i,n-i}^2 \cdot \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \sum_{i=2}^n w_{n-i+1} \cdot \mathbf{M}'_{n-i+1}}}.$$

Since we assume that all claims are settled after n development years, $\mathbf{Z}_{1,n} = \mathbf{Z}_{1,n}^{model} \equiv \hat{Z}_{1,n} = 0$ for all n .

We have

$$\begin{aligned}
\mathbf{Z}_{adj}^{model} &= (1 - \mathbf{a}_{sim}) \sum_{i=2}^n \hat{Z}_{i,n-i+1} + \mathbf{a}_{sim} \sum_{i=2}^n \mathbf{Z}_{i,n-i+1}^{model} \\
&= (1 - \mathbf{a}_{sim}) \sum_{i=2}^n (\hat{f}_{n-i+1} - 1) \cdot C_{i,n-i} \\
&\quad + \mathbf{a}_{sim} \sum_{i=2}^n \left(\hat{f}_{n-i+1} - \sigma_{n-i+1}^{sim} \cdot \mathbf{R}'_{n-i+1} + \frac{\sigma_{n-i+1}^{sim}}{\sqrt{C_{i,n-i}^\gamma}} \cdot \zeta'_{i,n-i+1} - 1 \right) \cdot C_{i,n-i} \\
&\sim \sum_{i=2}^n (\hat{f}_{n-i+1} - 1) \cdot C_{i,n-i} + \mathbf{a}_{sim} \sqrt{\sum_{i=2}^n C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) (\sigma_{n-i+1}^{sim})^2} \cdot \zeta'
\end{aligned}$$

with independent, standard normally distributed random variables ζ' and $\zeta'_{i,n-i+1}$ independent of \mathbf{M}'_{n-i+1} .

Hence,

$$\begin{aligned}
\mathbf{Z}_{adj}^{model} &\sim \sum_{i=2}^n (\hat{f}_{n-i+1} - 1) \cdot C_{i,n-i} + \sqrt{\frac{\sum_{i=2}^n \hat{\sigma}_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n w_{n-i+1} \mathbf{M}'_{n-i+1}}} \cdot \zeta' \\
&= \sum_{i=2}^n (\hat{f}_{n-i+1} - 1) \cdot C_{i,n-i} \\
&\quad + \sqrt{\frac{\sum_{i=2}^n \hat{\sigma}_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \sum_i \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \mathbf{M}'_{n-i+1}}} \cdot \zeta' \\
&= \sum_{i=2}^n (f_{n-i+1} - 1) \cdot C_{i,n-i} + \sum_{i=2}^n \sigma_{n-i+1} \cdot R_{n-i+1} \cdot C_{i,n-i} \\
&\quad + \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \cdot M_{n-i+1} \cdot \sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \mathbf{M}'_{n-i+1}}} \cdot \zeta'
\end{aligned}$$

where R_{n-i+1} is a realization of \mathbf{R}_{n-i+1} such that $\hat{f}_{n-i+1} = f_{n-i+1} + \sigma_{n-i+1} \cdot R_{n-i+1}$ and M_{n-i+1} with is a realization of the random variable \mathbf{M}_{n-i+1} such

that $\hat{\sigma}_{n-i+1}^2 = \sigma_{n-i+1}^2 \cdot M_{n-i+1}$.

We deduce that

$$\begin{aligned} \mathbf{Z}_{adj}^{model} &\sim \sum_{i=2}^n (f_{n-i+1} - 1) \cdot C_{i,n-i} + \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma}} \cdot \tilde{\zeta} \\ &+ \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \cdot M_{n-i+1} \cdot \sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M'_{n-i+1}}} \cdot \zeta' \end{aligned}$$

where $\tilde{\zeta} := \sum_i \sigma_{n-i+1} \cdot R_{n-i+1} \cdot C_{i,n-i} \cdot \left(\sqrt{\frac{\sum_i \sigma_{n-i+1}^2 C_{i,n-i}^2}{\sum_l C_{l,n-i}^\gamma}} \right)^{-1}$ is a realization of a standard normally distributed random variable $\tilde{\zeta}$ independent of both, ζ' and \mathbf{M}_i for all i .

With an independent standard normally distributed random variable ζ it follows that

$$\begin{aligned} F_{\mathbf{Z}_{adj}^{model}} \left(\sum_{i=2}^n \mathbf{Z}_i \right) &= F_{\mathbf{Z}_{adj}^{model}} \left(\sum_{i=2}^n (f_{n-i+1} - 1) C_{i,n-i} + \sigma_{n-i+1}^2 \sqrt{C_{i,n-i}^{2-\gamma}} \cdot \zeta_{i,n-i+1} \right) \\ &= F_{\mathbf{Z}_{adj}^{model}} \left(\sum_{i=2}^n (f_{n-i+1} - 1) C_{i,n-i} + \sqrt{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^{2-\gamma}} \cdot \zeta \right) \\ &= P \left(\sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma}} \cdot \tilde{\zeta} \right. \\ &\quad \left. + \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \mathbf{M}_{n-i+1} \sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) \mathbf{M}'_{n-i+1}}} \cdot \zeta' \right) \\ &\leq \sqrt{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^{2-\gamma}} \cdot \zeta \end{aligned}$$

This can be rewritten as

$$\begin{aligned}
& P \left(\sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M'_{n-i+1}}} \cdot \zeta' \right. \\
& \quad \left. - \sqrt{\frac{\sigma_{n-i+1}^2 C_{i,n-i}^2}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma}} \cdot \tilde{\zeta} + \sqrt{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^{2-\gamma}} \cdot \zeta \right) \\
& \leq \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M_{n-i+1}}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M'_{n-i+1}}} \\
& P \left(\sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M'_{n-i+1}}} \cdot \zeta' \right. \\
& \quad \left. \leq \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M_{n-i+1}}} \cdot \zeta \right) \\
& = F \sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M'_{n-i+1}}} \cdot \zeta' \left(\sqrt{\frac{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right)}{\sum_{i=2}^n \sigma_{n-i+1}^2 C_{i,n-i}^2 \left(\frac{1}{C_{i,n-i}^\gamma} + \frac{1}{\sum_{l=0}^{i-1} C_{l,n-i}^\gamma} \right) M_{n-i+1}}} \cdot \zeta \right).
\end{aligned}$$

Hence, $F_{\mathbf{Z}_{adj}^{model}}(\sum_i \mathbf{Z}_i)$ is uniformly distributed and the assertion follows. \square

B General procedure in Section 4

B.1 Example

Throughout Section 4 we use the following example: Consider a claims development triangle with $n = 10$. We assume that the starting values $C_{i,0}$ are normally distributed with mean $f_0 = 1,420,000$ EUR and standard deviation $\sigma_0 = 336,000$ EUR.

The “true” parameters f_k and σ_k are given by

	Development year k									
	1	2	3	4	5	6	7	8	9	10
f_k	1.5	1.2	1.12	1.07	1.04	1.02	1.01	1.005	1.002	1.0
$\sigma_k \cdot \sqrt{f_0}^{-\gamma}$	0.2	0.12	0.08	0.045	0.03	0.018	0.01	0.006	0.003	0.0

Table 4: Development factors and their standard deviation

Note that we assume all payments to be settled after 10 development years, i.e. $f_{10} = 1$ and $\sigma_{10} = 0$.

B.2 The general procedure to determine the probability of solvency

To determine the probability of solvency

$$P(\mathbf{X} \leq \text{SCR}(99.5\%; \mathbf{D}; M)) \quad (13)$$

experimentally we use the following general procedure based on a Monte-Carlo simulation. Fixing the “true” parameters f_k , σ_k we run through the following steps:

1. (Outer loop over s different random triangles) Using the normal model assumptions and given the parameters f_k and σ_k , $0 \leq k \leq n$, draw s different random development triangles D_j , $1 \leq j \leq s$.
2. For each triangle D_j , $1 \leq j \leq s$, do the following:
 - (a) (Simulation of the “true” claims development result \mathbf{X}) Using the normal model assumptions and the parameters f_k and σ_k draw random realizations of $\mathbf{Z}_{i,n-i+1}$, $1 \leq i \leq n$, representing the payments of the next business year. Estimating $\hat{R}_0(D_j)$ and $\hat{R}_1(D_j; \vec{\mathbf{Z}})$ as described in Subsection 2.2 using the deterministic chain-ladder method we get a realization x_j of

$$\mathbf{X} = \mathbf{X}(D_j, \vec{\mathbf{Z}}) = \hat{R}_1(D_j, \vec{\mathbf{Z}}) + \mathbf{Z} - \hat{R}_0(D_j)$$

representing the claims development loss of the next business year.

- (b) (Determination of the risk capital) Independently of x_j we then determine the SCR using a Monte-Carlo simulation with t scenarios. For each of the t scenarios we draw a realization from $\mathbf{X}^{model}(D_j)$ where

$$\mathbf{X}^{model}(D_j) := \begin{cases} \mathbf{X}_{without}^{model}(D_j) & \text{in Subsection 4.1,} \\ \mathbf{X}_{BT}^{model}(D) & \text{in Subsection 4.2,} \\ \hat{\mathbf{X}}_{adj}^{model}(D) & \text{in Subsection 4.3.} \end{cases}$$

We set the solvency capital requirement $\text{SCR} = \text{SCR}(D_j)$ equal to the empirical α -quantile determined by the simulation described above. It approximates the quantile $F_{\mathbf{X}^{model}}^{-1}(\alpha)$.

- (c) (Does the risk capital cover the loss?) We compare the realization x_j with $\text{SCR} = \text{SCR}(D_j)$.
3. (Determination of the probability of solvency) Count how many times we observe $x_j \leq \text{SCR}(D_j)$. The relative frequency approximates the probability (13).

For the calculations of the results presented in Section 4 we used $s = 100,000$ and $t = 10,000$ simulations.

Remark B.1. For the normal model it is theoretically possible that the chain ladder development factors become negative resulting in negative cumulative claims. In the rare cases where we observed negative factors we reset the factor equal to 1.0. Note that small factors correspond to small realizations of \mathbf{S} which do not effect the probability of solvency focusing on large realizations.

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